# Reduced-Order Modeling of Flexible Structures

Jayant V. Ramakrishnan,* S. Vittal Rao,† and Leslie R. Koval‡
*University of Missouri—Rolla, Rolla, Missouri*

**An alternate procedure for deriving a reduced-order model is presented. The Routh expansion method is used and preserves the original system impulse response energy. This procedure does not acquire knowledge of the system eigenvalues/eigenvectors and guarantees a stable reduced-order model if the original system is stable. The method is illustrated on a simply supported beam subjected to moment excitation. A comparison is made with reduced-order models obtained using other methods.**

## Introduction

FLEXIBLE structures are described by partial differential equations, and a common practice is to represent their dynamics via linear ordinary differential equations using the finite-element discretization technique. The finite-element model often is too large for control applications and, hence, reduced-order models are usually employed. A number of methods for deriving reduced-order models, such as parameter optimization methods, eigenvalue truncation, and component cost analysis (CCA) methods, are available and have been described in the literature.

The CCA method of Skelton et al.[1,2] requires computing the "costs" of each mode; the modes with the highest costs are retained in the reduced-order model. The modal cost function is based on eigenvalue, eigenvector, damping, sensor, and actuator information. The major drawback of CCA is that it requires the computation of the eigenvalues and eigenvectors of the structure. This can be quite a task in itself in the case of large finite-element models.

The present work considers an alternate model reduction technique, which preserves a percentage of the original system impulse response energy. The method used is the Routh expansion method. Hutton and Friedland[3] have proposed this method in the frequency domain for the single-input/single-output (SISO) systems and, later, Rao et al.[4] extended the procedure to the time domain by eliminating the reciprocal transformation. This simplification procedure requires no knowledge of the system eigenvalues/eigenvectors and guarantees a stable reduced-order model if the original system is stable. The method does not directly extend to multi-input/multi-output (MIMO) systems. Hwang and Guo[8] have presented a method for deriving reduced-order models for multivariable systems by using matrix Routh approximant methods. This procedure does not guarantee the stability of the reduced-order models. The multivariable Routh method has been used to derive a reduced-order model for a structure. It has been shown that, for a case of symmetrically located actuators, the reduced-order multivariable models are stable if the original system is stable.

In the present work, we consider a simply supported beam subjected to a moment excitation. Reduced-order models derived using the Routh methods are compared with those obtained using the CCA method and the exact Rayleigh-Ritz solution.

Received Jan. 5, 1987; revision received April 2, 1987. 

*Department of Mechanical Engineering. Student Member AIAA.
†Department of Electrical Engineering.
‡Professor, Department of Mechanical Engineering. Member AIAA.

## Structural Modeling by the Finite-Element Method

The structure chosen in this paper is a Bernoulli-Euler beam represented by the partial differential equation

$$\rho(r)\ddot{q}(r,t) + EI\, q^{iv}(r,t) = f(t)\,\delta(r - r_c) \tag{1}$$

$$y = q(r_0,t) \tag{2}$$

where $\rho(r)$ is the mass per unit length of the beam, $EI$ is the flexural rigidity of the beam, $\ddot{q}(r,t)$ represents partial differentiation with respect to time, and $q^{iv}(r,t)$ represents partial differentiation with respect to the space variable. The excitation is provided by a discrete moment applied at $r = r_c$. A displacement sensor is located at $r = r_o$. In the finite-element context, we choose a finite set of basis or shape functions, such that

$$q(r,t) = \sum_{i=1}^{j} \psi_i(r) p_i(t) \tag{3}$$

where $p_i(t)$ are generalized coordinates and $\psi_i(r)$ the shape functions. Then, by using standard techniques,[6] we obtain a set of finite ordinary differential equations of the form

$$[M]\,\{\ddot{p}\} + [K]\,\{p\} = \{F\}w(t) \tag{4}$$

$$y = \sum_i \psi_i(r_o)p(t) \tag{5}$$

where $M \in \mathbb{R}^{nq \times nq}$ is the structure mass matrix, $K \in \mathbb{R}^{nq \times nq}$ is the structure stiffness matrix, $F \in \mathbb{R}^{nq \times m}$ is the structure force influence matrix, and $w(t)$ describes the time variation of the external excitation.

The size and entries in the mass and stiffness matrices depend on the discretization of the beam. The output vector entries also depend on the chosen finite-element grid. The matrices generated using the finite-element technique are banded in nature. The elements of the mass and stiffness matrices are given by the following integrals[1,6]:

$$M_{ij} = \int_L \psi_i^T(r)\rho(r)\psi_j(r)\,\mathrm{d}r \tag{6}$$

$$K_{ij} = \int_L \left(\frac{\partial^2 \psi_i}{\partial r^2}\right)^T EI\left(\frac{\partial^2 \psi_j}{\partial r^2}\right)\mathrm{d}r \tag{7}$$

The force vector in the case of a discrete moment excitation is given by

$$F_i = \int_L \frac{\partial}{\partial r}\,[\psi_i^T(r)]\delta(r - r_c)\,\mathrm{d}r \tag{8}$$

Damping is introduced into all modes as viscous damping with a damping factor $\zeta$ of 0.005. The corresponding damping matrix is represented by $[C]$. The details on shape functions and the finite-element matrices are available in Refs. 1 and 6. In the state-space notation, these $nq$ second-order differential equations are expressed as $2 \times nq$ first-order differential equations written in matrix notation as

$$\{\dot{z}\} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \{z\} + \begin{Bmatrix} 0 \\ M^{-1}F \end{Bmatrix} T \tag{9}$$

where $T$ is external applied moment and $[z] = [p, \dot{p}]^T$.

The analytical solution of this problem using the separation of variables technique and the final expression for the time response are derived in Appendix A.

## Reduced-Order Modeling by the Routh Expansion Methods

A brief review of the Routh expansion method is presented in this section.

Given a system represented by

$$\dot{x} = Ax + bu, \qquad y = Dx \tag{10}$$

where $A \in \Re^{n \times n}$, $b \in \Re^{n \times 1}$, $D \in \Re^{p \times n}$ and assuming, without loss of generality, that the system is in a phase variable canonical form, there exists a transformation,[4]

$$v = Px \tag{11}$$

which transforms the system in Eq. (10) into the $\gamma - \delta$ canonical form

$$\dot{v} = Rv + \tilde{b}u, \qquad y = \tilde{D}v \tag{12}$$

where $R = PAP^{-1}$, $\tilde{b} = Pb$, and $\tilde{D} = DP^{-1}$. A procedure for deriving the transformation matrix $P$ is available in Ref. 4. The matrix $P$ is an upper triangular matrix and simplifies the computational effort in obtaining $P^{-1}$. For $n$ even (in structural applications this is always true), the matrices $R$, $\tilde{b}$, and $\tilde{D}$ are given by

$$R = \begin{bmatrix} 0 & \gamma_2 & 0 & \gamma_4 & 0 & \gamma_6 \cdots & \gamma_n \\ -\gamma_1 & -\gamma_2 & 0 & -\gamma_4 & 0 & \gamma_6 \cdots & -\gamma_n \\ 0 & 0 & 0 & \gamma_4 & 0 & \gamma_6 \cdots & \gamma_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -\gamma_1 & -\gamma_2 & -\gamma_3 & -\gamma_4 & 0 & -\gamma_6 \cdots & -\gamma_n \end{bmatrix} \tag{13}$$

$$\tilde{b} = [0 \quad 1 \quad 0 \ldots 1]^T \tag{14}$$

$$\tilde{D} = [\delta_1 \quad \delta_2 \ldots \delta_n] \tag{15}$$

The reduced-order model of dimension $nr$

$$\dot{z} = \tilde{F}z + Gu, \qquad y_m = D_1 z \tag{16}$$

is derived as follows:

Let

$$T_1 = [I_{nr} : 0]$$

Then

$$\tilde{F} = T_1 R T_1^T, \quad G = T_1 \tilde{b}, \quad \text{and} \quad D_1 = \tilde{D} T_1^T$$

This time-domain Routh expansion method can be used to derive even-ordered reduced-order models from even-ordered original system models. However, it is not possible to directly derive odd-ordered reduced-order models from even-ordered original system models. In this work, we are interested in representing a set of complex conjugate eigenvalues, so this restriction does not pose problems in the single-input/single-output case.

The impulse energy of the $k$th-order model is given by[5]

$$E_k = \sum_{i=1}^{k} \delta_i^2 / 2\gamma_i \tag{17}$$

By the very nature of its evaluation, $\gamma_i$ is always greater than zero. We then have

$$E_{k+1} = \sum_{i=1}^{k+1} \delta_i^2 / 2\gamma_i = E_k + \delta_{k+1}^2 / 2\gamma_{k+1} > E_k \tag{18}$$

In this method, the impulse energy of the lower-order models monotonically converges to that of the original system energy as the model order is increased, i.e.,

$$0 \le E_1 \le E_2 \le \ldots \le E_n \tag{19}$$

The expression for the impulse response energy and the measure of improvement of the system performance are given in Ref. 5. If $E_i$ represents the impulse energy of the $i$th-order model, then

$$\hat{E}_i = (E_n - E_i)/E_n \times 100\% \tag{20}$$

where $E_n$ is the original system impulse energy. The measure of improvement is then defined as

$$\mu_i = \hat{E}_{i-1} / \hat{E}_i \tag{21}$$

and the best choice for the model order can be defined as that value of $i$ for which $\mu_i$ is a maximum.

## Reduced-Order Structural Model

In this paper, the preceding Routh approximant reduced-order methodology is applied to a finite-element structural model. The procedure is demonstrated with the help of a numerical example. Consider the simply supported beam shown in Fig. 1. The actuating moment is applied at the left support and the displacement sensor is located at a distance of $0.35L$ from the left support, where $L$ is the length of the beam. For the finite-element discretization shown in Fig. 1, the mass

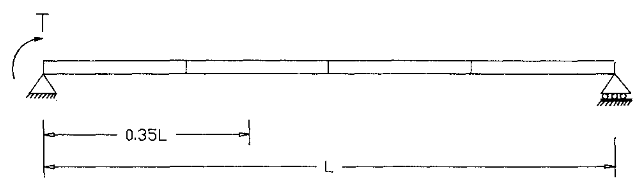


**Fig. 1 Simply supported beam.**

**Table 1 Undamped system frequencies**

| Mode $i$ | ω (Exact), rad/s | ω (Finite Element Method) rad/s |
|---|---|---|
| 1 | 0.0625 | 0.0625 |
| 2 | 0.2500 | 0.2510 |
| 3 | 0.5625 | 0.5728 |
| 4 | 1.0 | 1.1099 |
| 5 | 1.5625 | 1.7642 |
| 6 | 2.25 | 2.7899 |
| 7 | 3.0625 | 4.1796 |
| 8 | 4.0 | 5.0863 |

**Table 2 Measure of improvement**

| Model order | $E_i$ | $E_i$ | $\mu_i$ |
|---|---|---|---|
| 2 | 8031.0 | 25.27 | – |
| 4 | 10380.0 | 3.43 | 0.135 |
| 6 | 10696.0 | 0.483 | 7.1 |
| 8 | 10700.0 | 0.446 | 1.082 |
| 10 | 10741.0 | 0.065 | 6.861 |
| 12 | 10746.0 | 0.018 | 3.611 |
| 14 | 10747.0 | 0.009 | 2.0 |
| 16 | 10748.0 | 0.0 | – |

**Table 3 System frequencies**

| $i$ | Sixth-order | Sixteenth-order |
|---|---|---|
| 1,2 | $-0.0003 \pm j0.0625$ | $-0.0003 \pm j0.0625$ |
| 3,4 | $-0.0013 \pm j0.2510$ | $-0.0013 \pm j0.2510$ |
| 5,6 | $-0.0031 \pm j0.5791$ | $-0.0029 \pm j0.5728$ |
| 7,8 | – | $-0.0055 \pm j1.1099$ |
| 9,10 | – | $-0.0088 \pm j1.7642$ |
| 11,12 | – | $-0.0139 \pm j2.7898$ |
| 13,14 | – | $-0.0209 \pm j4.1796$ |
| 15,16 | – | $-0.0254 \pm j5.0862$ |

and stiffness matrices are of eighth-order, and we get a sixteenth-order state-space realization. It should be noted that we are interested in even-ordered reduced-order models. The values of the different constants are $L = 4\pi$, $EI = \rho$, $\rho = 2/L$, and $\zeta = 0.005$. The undamped system frequencies and their exact values are tabulated in Table 1.

The Routh expansion method is used to derive lower-order models for this sixteenth-order system representation. The impulse energies of various reduced-order models and the measure of improvement are given in Table 2.

The maximum value of $\mu_i$ occurs for $i = 6$. Thus, the sixth-order simplified model is the best choice for the sixteenth-order state-space representation. The damped eigenvalues of the sixth-order model and the sixteenth-order representation are given in Table 3 for comparison purposes. The evaluation of the eigenvalues of the sixteenth-order system representation is not required for deriving reduced-order modes.

It is obvious that the eigenvalues for the reduced-order model are not a subset of the original system eigenvalues. They are, in fact, a weighted sum of the original system eigenvalues, weighted such that the impulse energy is very close to that of the original system energy. The transient response of the sixth-order model, the sixteenth-order representation, and the exact solution are plotted in Fig. 2. The graph shows that the Routh approximant model matches the exact response very well.

The sixth-order Routh model can also be compared with the reduced-order model derived using the CCA procedure. According to the results in Ref. 1, the first three modes in order of their cost are modes 1, 2, and 4. A sixth-order model derived using the criterion is compared with the sixth-order Routh model in Fig. 3. Again, the responses match very well. It should be pointed out, that the Routh method proposed in this paper is computationally less expensive than the CCA method.

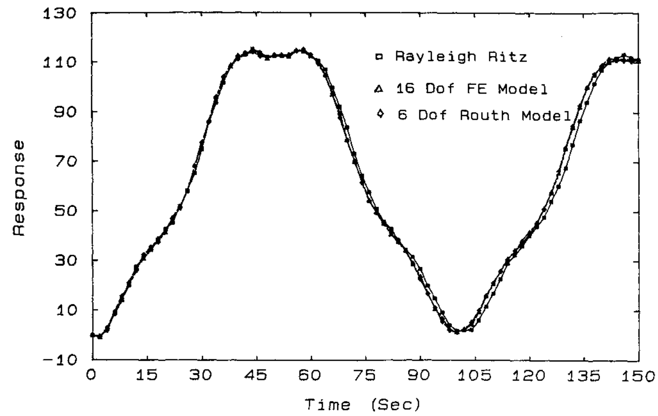


**Fig. 2 Comparison of system response.**

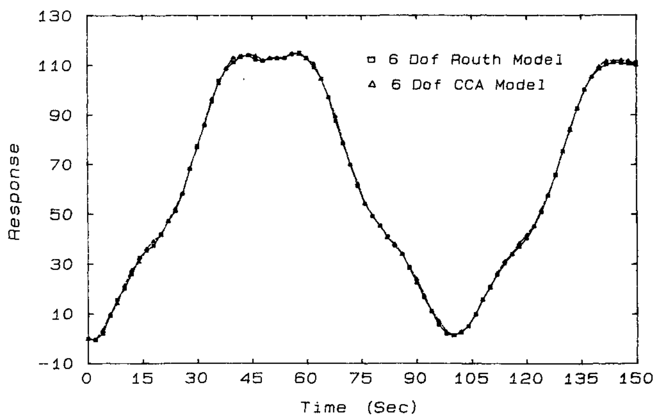


**Fig. 3 Comparison of system response (Routh and CCA model).**

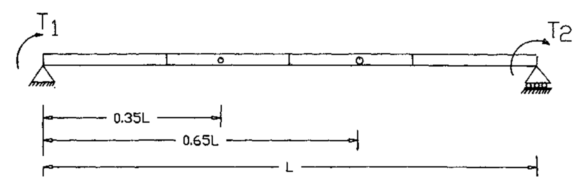


**Fig. 4 Geometry of the multivariable system.**

## Multi-Input/Multi-Output Case

In this section, a procedure is presented to derive a reduced-order model for a MIMO system.

The Routh approximant method in the multivariable domain is not a direct extension of the SISO method. The method suffers from a couple of limitations. First, there does not seem to be a direct analog of the Routh criterion for scalar polynomials as far as matrix polynomials are concerned. Thus, although the method is based on a "matrix Routh criterion," stability aspects are not addressed directly. The order of the system must be restricted to integral multiples of the number of inputs. Further, the restriction of reducing from even to even order in the time domain leads to less flexibility in the choice of lower-order models. A brief review of the multivariable Routh approximant method is given subsequently.

Given an $m$-input $p$-output system described by a state equation in general coordinates,

$$\dot{x} = Ax + Bu, \qquad y = Dx \tag{22}$$

where $x \in \Re^n$, $y \in \Re^p$, $u \in \Re^m$, $A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$, $D \in \Re^{p \times m}$, and $n = r_1 m$ ($r_1$ integer), provided the pair $(A,B)$ has a controllability index of $r_1$, it is possible to transform the system in Eq. (21) to a block controller form:

$$\dot{x}_c = A_c x_c + B_c u, \qquad y = D_c x_c \tag{23}$$

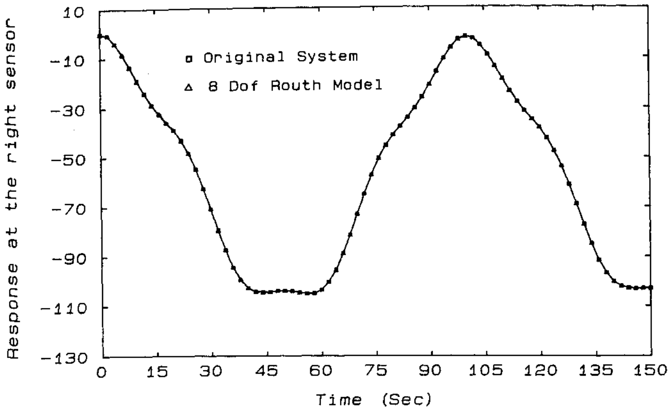

**Fig. 5 Right sensor response (moment at right actuator).**

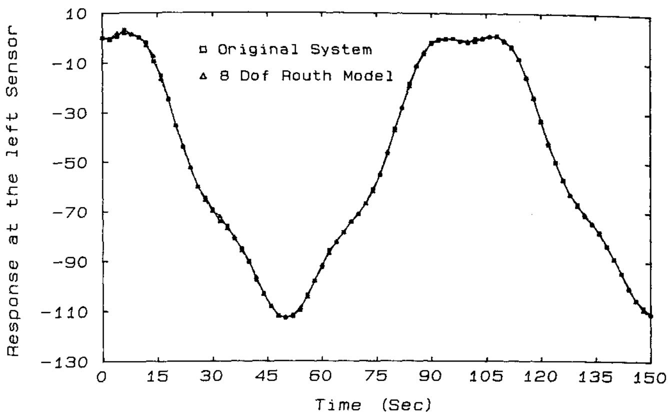

**Fig. 6 Left sensor response (moment at right actuator).**

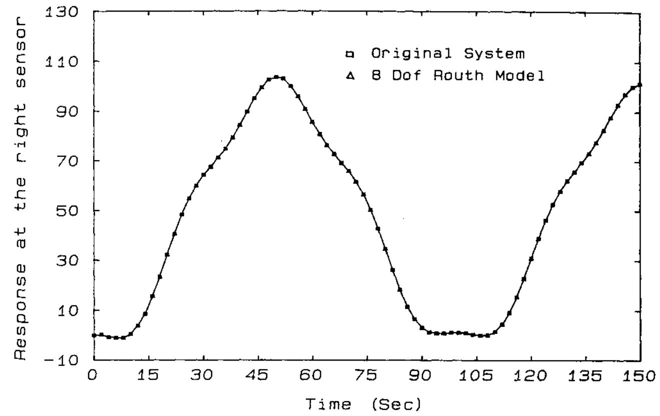

**Fig. 7 Right sensor response (moment at left actuator).**

The transformation matrix is given by

$$T_c = \begin{bmatrix} T_{c1} \\ T_{c1}A \\ \cdots \\ T_{c1}A^{r_1 - 1} \end{bmatrix};$$

$$T_{c1} = [O_m O_m \ldots I_m]\,[B\; AB \ldots A^{r_1-1}B]^{-1} \tag{24}$$

The matrices $A_c$, $B_c$, and $D_c$ have the form

$$A_c = T_c\, A\, T_c^{-1} = \begin{bmatrix} O_m & I_m & O_m \ldots & O_m \\ O_m & O_m & I_m \ldots & O_m \\ \cdot & & & \cdot \\ O_m & \ldots\ldots\ldots & & I_m \\ -A_{cr_1} & -A_{c(r_1-1)} & & -A_{c1} \end{bmatrix}$$

$$B_c = T_c\, B = [O_m\; O_m \ldots I_m]$$

$$D_c = D\, T_c^{-1} = [D_{cr_1}\, D_{c(r_1-1)} \ldots D_{c1}] \tag{25}$$

where $O_m$ is an $m \times m$ null matrix, $I_m$ an $m \times m$ identity matrix, $A_{ci} \in \Re^{m \times m}$ $A_{c0} = I_m$, and $C_c \in \Re^{p \times m}$. The right matrix fraction description of the transfer function corresponding to the system in Eq. (22) is

$$H(s) = N_R(s)\, D_R^{-1}(s) \tag{26}$$

where

$$N_R(s) = D_{c1}\, s^{r_1-1} + D_{c2}\, s^{r_1-2} + \ldots + D_{cr_1}$$

$$D_R(s) = I_m s_1^r + A_{c1}\, s^{r_1-1} + \ldots + A_{cr_1} \tag{27}$$

More details on this transformation can be found in Ref. 7.

Based on the low-frequency approximation, the matrix Routh table (Appendix B) can be constructed. The entries in the table are $m \times m$ matrices and, using the results in Ref. 8, a transformation matrix $P$, which transforms the system in Eq. (22) to the matrix canonical form,

$$\dot{v} = Rv + \bar{B}u, \qquad y = \bar{D}v \tag{28}$$

where $R = PA_cP^{-1}$, $\bar{B} = PB_c$, and $\bar{D} = D_cP^{-1}$.

The reduced-order model is derived as before with

$$T_1 = [I_{nr} : 0] \tag{29}$$

and the reduced-order model

$$\dot{z} = \bar{F}z + Gu, \qquad y_m = D_1 z \tag{30}$$

**Table 4 MIMO system frequencies**

| $i$ | Eighth-order | Sixteenth-order |
|---|---|---|
| 1,2 | $-0.0003 \pm j0.0625$ | $-0.0003 \pm j0.0625$ |
| 3,4 | $-0.0013 \pm j0.2510$ | $-0.0013 \pm j0.2510$ |
| 5,6 | $-0.0059 \pm j0.6822$ | $-0.0029 \pm j0.5728$ |
| 7,8 | $-0.0139 \pm j1.3823$ | $-0.0055 \pm j1.1099$ |
| 9,10 | – | $-0.0088 \pm j1.7642$ |
| 11,12 | – | $-0.0139 \pm j2.7898$ |
| 13,14 | – | $-0.0209 \pm j4.1796$ |
| 15,16 | – | $-0.0254 \pm j5.0862$ |

$$\bar{F} = T_1 R T_1^T, \quad G = T_1\bar{B}, \quad \text{and} \quad D_1 = \bar{D}T_1^T \tag{31}$$

In the multivariable case, the reduced-order models are not guaranteed stable unless the computed matrix $\Gamma$ coefficients are all symmetric and positive definite.[9]

## Multivariable Reduced-Order Structural Model

The matrix Routh method is applied to the beam model shown in Fig. 4. The finite-element discretization procedure is unaltered, and the mass, stiffness, and damping matrices remain the same. Actuating moments are applied at the left and right supports, and displacement sensors are located at $r_0 = 0.35L$ and $0.65L$. The discretization scheme in Fig. 4 produces a sixteenth-order system matrix $A$. The values of the different parameters are identical to the SISO example. The matrix Routh criterion is used to derive an eighth-order model. In this case, the matrix $\Gamma$ coefficients all turn out to be symmetric and positive definite. Hence, the reduced-order models are guaranteed stable. Further investigation shows that, when the actu-

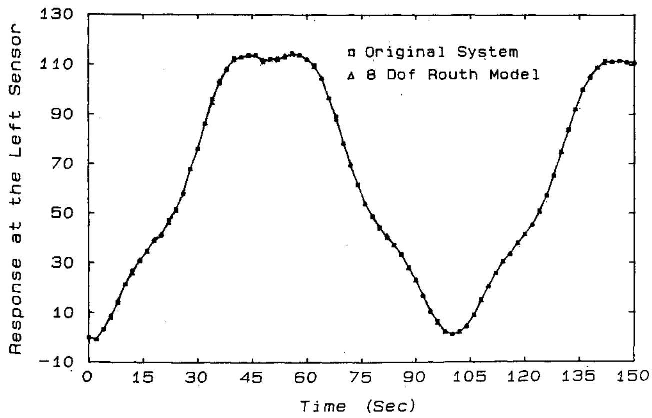


**Fig. 8 Left sensor response (moment at left actuator).**

ators are symmetrically located, this will always be true. The damped eigenvalues of the eighth-order model and sixteenth-order system are tabulated in Table 4.

It can be seen that the eigenvalues of the reduced-order scalar system are different from the reduced-order multivariable system eigenvalues. This is not entirely surprising, as the algorithms are a little different in the two cases. For the purpose of generating a transient response, the input moments have been modeled as step inputs, and the corresponding response is plotted. The results are shown in Figs. 5–8 and show that the response of the eighth-order model tracks the original system response very well.

## Conclusions

We have applied the Routh expansion technique to derive reduced-order models of flexible structures. The method is powerful in that it preserves stability and a significant percentage of the original system response energy. Further, it is computationally elegant in that it does not require the eigenanalysis of the system. The method has a couple of drawbacks as applied to multivariable systems. Stability of the multivariable lower-order models is not guaranteed in all cases. However, for symmetrically located actuators, stability of the lower-order model is guaranteed. Nevertheless, it is quite possible that these drawbacks can be eliminated or made less restrictive. The comparison of the reduced-order model response with the exact solution shows that the method has good accuracy and considerable promise. This method can be conveniently extended to reduce more complicated structures to computationally amenable dimensions for control simulation work. Currently, we are investigating the modeling and control of complex multi-input/multi-output structures using this technique.

## Appendix A

$$EI\, q^{iv}(r,t) + \rho(r)\, \ddot{q}(r,t) = f(t)\, \delta(r - r_c) \tag{A1}$$

Expanding $q(r,t)$ in terms of its normal modes

$$q(r,t) = \sum_i \phi_i(r)\, p_i(t) \tag{A2}$$

For a simply supported beam,

$$\phi_i(r) = \sin(i\pi r/L) \tag{A3}$$

and

$$\phi_i(r=0) = \phi_i''(r=0) = \phi_i(r=L) = \phi_i''(r=L) = 0 \tag{A4}$$

Applying the variational formulation on Eq. (A1),[6] and simplifying (for $p$ = constant),

$$\sum \left[\ddot{p}_i(t) + \frac{EI}{p(r)}\frac{i^4\pi^4}{L^4}p_i(t)\right] = \int_L \frac{f(t)\delta(r-r_c)}{p(r)} \sin\frac{i\pi r}{L}\,\mathrm{d}r \tag{A5}$$

Denoting

$$\omega_i^2 = \frac{EI}{p(r)}\frac{i^4\pi^4}{L^4} \tag{A6}$$

and introducing damping and evaluating the force term, we have

$$\sum_i \left[\ddot{p}_i(t) + 2\,\zeta\,\omega_i\,\dot{p}_i(t) + \omega_i^2\, p_i(t) = \frac{T}{p(r)}\frac{2i\pi}{L^2}\right] \tag{A7}$$

Substituting values for $L$, $\rho$, and $EI$ and solving Eq. (A7), we get

$$q(r_o,t) = \sum_i \sin(ir_o/L)\left\{2\exp[-\zeta(i/4)^2t]\left[-\frac{1}{2}(4/i)^3\right.\right.$$
$$\left.\left.(\cos 0.99\,(i/4)^2\,t + \zeta\sin 0.99\,(i/4)^2 t)\right] + (4/i)^3\right\} \tag{A8}$$

## Appendix B

The matrix characteristic equation of the MIMO system is

$$D_R(s) = 0 \tag{B1}$$

The low-frequency approximation is obtained by setting

$$(1/s)D_R(1/s) = 0 \tag{B2}$$

or

$$A_{cr_1}s^r + A_{c(r_1-1)}\,s^{(r-1)} + \ldots + I_m = 0 \tag{B3}$$

Define

$$A_{1,1} = A_{cr_1},\ A_{1,2} = A_{c(r_1-2)}, \ldots$$

and

$$A_{2,1} = A_{c(r_1-1)},\ A_{2,2} = A_{c(r_1-3)}, \ldots$$

The matrix Routh array is

$$\begin{array}{llll}
A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4}\ldots \\
A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4}\ldots \\
A_{3,1} & A_{3,2} & A_{3,3}\ldots & \\
A_{4,1} & A_{4,2} & A_{4,3}\ldots & \\
A_{5,1} & A_{5,2}\ldots & & \\
A_{6,1} & A_{6,2}\ldots & & \\
A_{7,1}\ldots & & & \\
A_{n,1} & & & \\
A_{n+1,1} & & &
\end{array}$$

where

$$A_{i,j} = A_{i-1,j+1} - \Gamma_{i-2}A_{i-1,j+1}$$

$$j = 1,2,\ldots,[(r_1+3-j)/2]$$

$$i = 3,4,\ldots,r_1+1$$

$$\Gamma_i = A_{i,1}A_{i+1,1}^{-1}, \quad i = 1,2,\ldots,r_1$$

$$\det(A_{i+1,1}) \neq 0$$

$$P = \begin{bmatrix} A_{2,1} & O_m & A_{2,2} & O_m & \ldots & O_m \\ O_m & A_{3,1} & O_m & A_{3,2} & \ldots & I_m \\ O_m & O_m & A_{4,1} & O_m & \ldots & O_m \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & O_m & A_{n,1} & O_m \\ O_m & O_m & \ldots & \ldots & \ldots & I_m \end{bmatrix} \quad \text{for } n \text{ even}$$

## References

[1]Skelton, R. E. and Hu, A., "Modeling and Control of Large Flexible Structures," First European Workshop on the Real Time Control of Large Scale Systems, Univ. of Patras, July 1984.

[2]Skelton, R. E., "Control Design of Flexible Spacecraft," Chap. 8, *Theory and Applications of Optimal Control in Aerospace Systems,* edited by P. Kant, AGARD Pub. 251, KBN 92-835-1391-6, July 1981. (Distributed in the United States by NASA, Langley Field, VA 23365, Attn: Report Distribution and Storage Unit.)

[3]Hutton, M. F. and Friedland, B., "Routh Approximations for Reducing the Order of Linear, Time-Invariant Systems," *IEEE Transactions on Automatic Control,* Vol. AC-20, June 1975.

[4]Rao, A. S., Lamba, S. S., and Rao, S. V., "Routh-Approximant Time-Domain Reduced-Order Models for Single-Input Single-Output Systems," *Proceedings of the IEEE,* Vol. 125, Oct. 1978.

[5]Subbarao, A., Lamba, S. S., and Rao, S. V., "Choice of Model Order," *Proceedings of Large Engineering Systems 3,* July 1980.

[6]Zienkiewicz, O. C., *The Finite Element Method in Engineering Science,* McGraw-Hill, New York, 1976.

[7]Shieh, L. S. and Tsay, Y. T., "Block Modal Matrices and Their Applications to Multivariable Control Systems," *Proceedings of the IEE,* Vol. 129, Pt. D, March 1982.

[8]Hwang, C. and Guo, T.-Y., "Matrix Routh-Approximant Reduced-Order Modelling for Multivariable Systems," *International Journal of Systems Science,* Vol. 16, No. 6, 1985, pp. 697–712.

[9]Shieh, L. S. and Sacheti, S., "A Matrix in the Block Schwarz Form and the Stability of Matrix Polynomials," *International Journal of Control,* Vol. 37, 1978, p. 245.